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# Cognitive concept learning from incomplete information

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**Abstract** Cognitive concept learning is to learn concepts from a given clue by simulating human thought processes including perception, attention and thinking. In recent years, it has attracted much attention from the communities of formal concept analysis, cognitive computing and granular computing. However, the classical cognitive concept learning approaches are not suitable for incomplete information. Motivated by this problem, this study mainly focuses on cognitive concept learning from incomplete information. Specifically, we put forward a pair of approximate cognitive operators to derive concepts from incomplete information. Then, we propose an approximate cognitive computing system to perform the transformation between granular concepts as incomplete information is updated periodically. Moreover, cognitive processes are simulated based on three types of similarities. Finally, numerical experiments are conducted to evaluate the proposed cognitive concept learning methods.

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## 1 Introduction

Cognitive computing is viewed as the development of computer systems modeled on the human brain [37]. As far as we know, such kind of computing is to simulate human thought processes by a computer such as thinking, learning, perception and attention. Cognitive learning, as a useful mathematical tool for the realization of cognitive computing, is considered as the function of simulating cognitive processes including the operations of learning, thinking and remembering something. With the development of cognitive learning, it has absorbed many effective methods from psychology, information theory and mathematics [36].

As is well known, a concept is the basic unit of human cognition [39]. Currently, many kinds of concepts [18, 34, 45, 46] have been developed to meet different requirements in the real world. It should be pointed out that these concepts have been applied in many fields such as rough analysis, rule acquisition and knowledge discovery [2, 4, 5, 12, 16, 19, 25, 27, 28, 38]. Moreover, granular computing [21–24, 41, 44, 48, 49] has been incorporated into concept learning to improve the learning efficiency [10, 20, 40].

Cognitive concept learning is to learn concepts from a given clue by simulating human thought processes. It was firstly investigated by Zhang and Xu [51], Wang [36] and Yao [47]. In recent years, there has been a growing interest

on this topic. For instance, Xu et al. [43] discussed how to obtain sufficient and necessary information granules from an arbitrary information granule. Li et al. [14] put forward three cognitive concept learning methods from the perspectives of philosophy and cognitive psychology, and they [11] also designed a cognitive concept learning framework for big data. Moreover, the theory of three-way decisions has been incorporated into cognitive concept learning [15] as well. In addition, Aswani Kumar et al. [3] analyzed the concept learning mechanism of exploring cognitive functionalities of bidirectional associative memory. Very recently, Xu and Li [42] reconsidered the issue of cognitive concept learning under a fuzzy environment.

However, with the deepening of the research on cognitive concept learning, it is found that the classical cognitive concept learning methods are restrictive for many applications. For example, they are not suitable for incomplete information which is often encountered in the real world [13]. The main theme of this paper is to address this issue. Specifically, a pair of approximate cognitive operators is proposed to learn concepts from incomplete information. Then, an approximate cognitive computing system is established to perform the transformation between granular concepts as incomplete information is updated periodically. In addition, cognitive processes are simulated based on three types of similarities. Finally, we conduct some numerical experiments to evaluate the proposed cognitive concept learning methods.

The remainder of this paper is organized as follows. Section 2 analyzes cognitive mechanism of learning approximate concepts from incomplete information.

**Table 1** A SARS dataset

| Patient | Fever | Cough | Headache | Difficulty breathing |
|---------|-------|-------|----------|----------------------|
| 1 | Yes | Yes | No | Yes |
| 2 | No | Yes | No | Yes |
| 3 | No | No | Yes | No |
| 4 | Yes | No | Yes | No |

Section 3 establishes an approximate cognitive computing system. Section 4 simulates cognitive processes based on similarity. Section 5 conducts some numerical experiments to assess the proposed concept learning methods. The paper is concluded in Sect. 6 with a brief summary and an outlook for further research.

## 2 Cognitive mechanism of learning approximate concepts from incomplete information

First of all, let us begin this section with an example.

*Example 1* Table 1 depicts a dataset of four patients who suffer from *severe acute respiratory syndrome* (SARS) [14]. In the table, *Fever*, *Cough*, *Headache* and *Difficulty breathing* are four symptoms which were observed from the current patients.

As time goes on, however, there will appear more SARS patients from whom new symptoms could be found such as *Diarrhea*, *Muscle aches*, *Nausea and vomiting*. Suppose that the updating information is shown in Table 2.

From Table 2, it is noticed that the values of patients 1, 2, 3 and 4 under the new symptoms (Diarrhea, Muscle aches, Nausea and vomiting) were missing. This is not surprising because the previous patients may have gone and could not be contacted when it is able to test these symptoms by new technologies.

Unfortunately, for such a SARS dataset with incomplete information, the classical cognitive concept learning methods will become invalid since missing values are not allowed for them. Therefore, it is necessary to develop some new cognitive concept learning methods which are able to handle incomplete information.

To facilitate our subsequent discussions, let $U$ be a nonempty object set and $A$ be an attribute set. Moreover, $2^U$ and $2^A$ are used to denote the power sets of $U$ and $A$, respectively. Then, a partial order relation is defined on the Cartesian product $2^A \times 2^A$ as

**Table 2** A SARS dataset with information updating on patients and symptoms

| Patient | Fever | Cough | Headache | Difficulty breathing | Diarrhea | Muscle aches | Nausea and vomiting |
|---------|-------|-------|----------|----------------------|----------|--------------|---------------------|
| 1 | Yes | Yes | No | Yes | ? | ? | ? |
| 2 | No | Yes | No | Yes | ? | ? | ? |
| 3 | No | No | Yes | No | ? | ? | ? |
| 4 | Yes | No | Yes | No | ? | ? | ? |
| 5 | Yes | No | Yes | No | Yes | No | No |
| 6 | No | Yes | No | No | No | Yes | No |
| 7 | Yes | No | No | Yes | No | No | Yes |
| 8 | No | Yes | No | No | No | Yes | Yes |
| 9 | No | No | Yes | No | Yes | No | No |

$$(B_1, C_1) \leq (B_2, C_2) \Longleftrightarrow B_1 \subseteq B_2 \text{ and } C_1 \subseteq C_2 \quad (1)$$

and the intersection and union are respectively given by

$$(B_1, C_1) \cap (B_2, C_2) = (B_1 \cap B_2, C_1 \cap C_2),$$
$$(B_1, C_1) \cup (B_2, C_2) = (B_1 \cup B_2, C_1 \cup C_2). \quad (2)$$

**Definition 1** Let $f : 2^U \to 2^A \times 2^A$ and $g : 2^A \times 2^A \to 2^U$ be two mappings. For $X_1, X_2 \subseteq U$ and $(B, C) \in 2^A \times 2^A$, if the following properties hold:

(i)   $X_1 \subseteq X_2 \Rightarrow f(X_1) \geq f(X_2)$,
(ii)  $f(X_1 \cup X_2) \geq f(X_1) \cap f(X_2)$,
(iii) $g(B, C) = \{x \in U | (B, C) \leq f(\{x\})\}$,

then $f$ and $g$ are called approximate-concept-forming cognitive operators (or simply approximate cognitive operators).

From Definition 1, it is easy to obtain the following properties:

$$f(X_1 \cup X_2) = f(X_1) \cap f(X_2),$$
$$g((B_1, C_1) \cup (B_2, C_2)) = g(B_1, C_1) \cap g(B_2, C_2). \quad (3)$$

**Theorem 1** *Let $f$ and $g$ be approximate cognitive operators. Then, they form a Galois connection between $2^U$ and $2^A \times 2^A$.*

*Proof* From (iii) of Definition 1, it is easy to get

$$(B_1, C_1) \leq (B_2, C_2) \Longrightarrow g(B_1, C_1) \supseteq g(B_2, C_2). \quad (4)$$

Note that $g(f(X)) = \{x \in U | f(X) \leq f(\{x\})\}$, and any $x \in X$ satisfies $f(X) \leq f(\{x\})$ based on (i) of Definition 1. So, we obtain

$$X \subseteq g(f(X)). \quad (5)$$

Suppose that $X_0$ is the objects satisfying $(B, C) \leq f(\{x\})$. Then, by (iii) of Definition 1 and Eq. (3), it follows

$$\begin{aligned} f(g(B, C)) &= f(\{x \in U | (B, C) \leq f(\{x\})\}) \\ &= f(X_0) \\ &= \bigcap_{x \in X_0} f(\{x\}) \\ &\geq (B, C). \end{aligned} \quad (6)$$

Combining (*i*) of Definition 1 with Eqs. (4), (5) and (6), we conclude that $f$ and $g$ form a Galois connection between $2^U$ and $2^A \times 2^A$.                                       □

**Definition 2** Let $f$ and $g$ be approximate cognitive operators. For $X \subseteq U$ and $(B, C) \in 2^A \times 2^A$, if $f(X) = (B, C)$ and $g(B, C) = X$, then $(X, (B, C))$ is called a concept of the approximate cognitive operators $f$ and $g$ (or simply approximate cognitive concept). In this case, $X$ and $(B, C)$ are referred to as the extent and intent of the approximate cognitive concept $(X, (B, C))$, respectively.

Moreover, the infimum ($\bigwedge$) and supremum ($\bigvee$) of a set of approximate cognitive concepts $\{(X_t, (B_t, C_t)) \mid t \in T\}$ are respectively given by

$$\bigwedge_{t \in T}(X_t, (B_t, C_t)) = \left(\bigcap_{t \in T} X_t, fg\left(\bigcup_{t \in T}(B_t, C_t)\right)\right),$$
$$\bigvee_{t \in T}(X_t, (B_t, C_t)) = \left(gf\left(\bigcup_{t \in T} X_t\right), \bigcap_{t \in T}(B_t, C_t)\right), \quad (7)$$

where $fg(\cdot)$ and $gf(\cdot)$ denote the composite mappings $f(g(\cdot))$ and $g(f(\cdot))$, respectively.

In what follows, we show how to derive approximate cognitive concepts from incomplete information. Before embarking on this issue, we need to construct a special pair of approximate cognitive operators $f$ and $g$.

Let $I$ be a "three-valued" mapping from $U \times A$ to $\{1, 0, ?\}$ for describing incomplete information. Specifically, $I(x, a) = 1$ indicates that the object $x$ possesses the attribute $a$, $I(x, a) = 0$ indicates that the object $x$ does not possess the attribute $a$, and $I(x, a) = ?$ indicates that it is unknown whether the object $x$ possesses the attribute $a$. Then, for any $X \subseteq U$, two mappings $\pi(\cdot)$ and $\theta(\cdot)$ are respectively defined as

$$\pi(X) = \{a \in A | \forall x \in X, I(x, a) = 1\},$$
$$\theta(X) = \{a \in A | \forall x \in X, I(x, a) = 1 \text{ or } I(x, a) = ?\}. \quad (8)$$

In a similar manner, for any $B \subseteq A$, we define

$$\pi(B) = \{x \in U | \forall a \in B, I(x, a) = 1\},$$
$$\theta(B) = \{x \in U | \forall a \in B, I(x, a) = 1 \text{ or } I(x, a) = ?\}. \quad (9)$$

Furthermore, $\pi(\cdot)$ and $\theta(\cdot)$ are used to construct a special pair of approximate cognitive operators $f$ and $g$:

$$\begin{aligned} f(X) &= (\pi(X), \theta(X)), \\ g(B, C) &= \{x \in U | (B, C) \leq f(\{x\})\}. \end{aligned} \quad (10)$$

*Example 2* Continued with Example 1. For convenience, in Table 2, $a$, $b$, $c$, $d$, $e$, $f$ and $g$ are used to represent the symptoms *Fever*, *Cough*, *Headache*, *Difficulty breathing*, *Diarrhea*, *Muscle aches*, and *Nausea and vomiting*, respectively. Moreover, number 1 is used to replace "Yes", number 0 is used to replace "No", and the question mark "?" is reserved. Then, combining Eqs. (7) and (8) with Definition 2, we can verify that $(45, (ac, ace))$ is an approximate cognitive concept, where 45, $ac$ and $ace$ are the abbreviations of $\{4, 5\}$, $\{a, c\}$ and $\{a, c, e\}$, respectively.

# 3 An approximate cognitive computing system

In the real world, objects and attributes are updated dynamically, and they often accompany with incomplete information. So, the currently obtained approximate

cognitive concepts need to be updated from time to time. To address this problem, an approximate cognitive computing system is proposed in this section.

## 3.1 Information granules and granular concepts

We first put forward the notion of information granules of approximate cognitive operators $f$ and $g$. For brevity, hereinafter, $f(\{x\})$ and $\pi(\{a\})$ are rewritten as $f(x)$ and $\pi(a)$, and $\pi(\pi(\cdot))$, $\theta(\pi(\cdot))$ and $f(\pi(\cdot))$ are abbreviated as $\pi\pi(\cdot)$, $\theta\pi(\cdot)$ and $f\pi(\cdot)$, respectively.

**Definition 3** Let $f$ and $g$ be approximate cognitive operators. Then, $f^G = \{\{x\} \mapsto f(x) | x \in U\}$ and $g^G = \{f\pi(a) \mapsto gf\pi(a) | a \in A\}$ are called information granules of $f$ and $g$, respectively.

The main purpose of introducing information granules is to simplify the description of approximate cognitive operators. In other words, basic information granules have the same ability to describe the mapping information under consideration. See the following theorem for the details.

**Theorem 2** Let $f^G$ and $g^G$ be the information granules of approximate cognitive operators $f$ and $g$. For $X \subseteq U$ and $(B, C) \in 2^A \times 2^A$ with $B = \bigcup_{a \in B} \pi\pi(a)$ and $C = \bigcup_{a \in B} \theta\pi(a)$, we have

$$f(X) = \bigcap_{x \in X} f^G(x),$$
$$g(B, C) = \bigcap_{a \in B} g^G f\pi(a). \tag{11}$$

*Proof* By Eq. (3), it follows $f(X) = \bigcap_{x \in X} f^G(x)$ directly. Besides, based on Eqs. (8), (9) and (10), we get

$$\begin{aligned} g(B, C) &= g\left(\bigcup_{a \in B} \pi\pi(a), \bigcup_{a \in B} \theta\pi(a)\right) \\ &= g\left(\bigcup_{a \in B} (\pi\pi(a), \theta\pi(a))\right) \\ &= g\left(\bigcup_{a \in B} f\pi(a)\right) \\ &= \bigcap_{a \in B} g^G f\pi(a). \end{aligned}$$

$\square$

**Theorem 3** Let $f$ and $g$ be approximate cognitive operators. Then, for any $x \in U$ and $a \in A$, both $(gf^G(x), f^G(x))$ and $(g^G f\pi(a), f\pi(a))$ are approximate cognitive concepts.

*Proof* The proof is obvious from Definition 2. $\square$

**Definition 4** Let $f$ and $g$ be approximate cognitive operators. For any $x \in U$ and $a \in A$, we say that $(gf^G(x), f^G(x))$ and $(g^G f\pi(a), f\pi(a))$ are granular concepts.

It can be known from Definition 3 and Theorem 2 that granular concepts can easily be computed by the information granules of $f$ and $g$.

**Theorem 4** Let $f$ and $g$ be approximate cognitive operators, and $(X, (B, C))$ be an approximate cognitive concept. Then,

(i) $(X, (B, C)) = \bigvee_{x \in X} (gf^G(x), f^G(x));$

(ii) $(X, (B, C)) = \bigwedge_{a \in B} (g^G f\pi(a), f\pi(a))$ if for any $c \in C \backslash B$, there exists $b \in B$ such that $\pi(b) \subseteq \theta(c)$.

*Proof* Note that $(B, C) = f(X) = f\left(\bigcup_{x \in X} \{x\}\right) = \bigcap_{x \in X} f^G(x)$. Based on Eq. (7), we know that $(X, (B, C))$ and $\bigvee_{x \in X} (gf^G(x), f^G(x))$ have the same intent. As a result, they are the same approximate cognitive concept. That is, (i) is proved. The remainder is to prove (ii).

Firstly, we show $B = \bigcup_{a \in B} \pi\pi(a)$. Note that $B \subseteq \bigcup_{a \in B} \pi\pi(a)$ due to $a \in \pi\pi(a)$. For any $b \in \bigcup_{a \in B} \pi\pi(a)$, there exists $a_0 \in B$ such that $b \in \pi\pi(a_0)$. Since $a_0 \in B \Rightarrow \pi(a_0) \supseteq \pi(B) \supseteq X \Rightarrow \pi\pi(a_0) \subseteq \pi(X) = B$, it follows $b \in B$, yielding $\bigcup_{a \in B} \pi\pi(a) \subseteq B$. To sum up, $B = \bigcup_{a \in B} \pi\pi(a)$ is true.

Secondly, we prove $C = \bigcup_{a \in B} \theta\pi(a)$. On one hand, for any $c \in C$, if $c \in B$, then $c \in \bigcup_{a \in B} \theta\pi(a)$ due to $a \in \theta\pi(a)$; otherwise, $c \in C \backslash B$. In this case, based on the assumption that there exists $b \in B$ such that $\pi(b) \subseteq \theta(c)$, we have $c \in \theta\pi(b)$. That is, $c \in \bigcup_{a \in B} \theta\pi(a)$ is also true. As a result, $C \subseteq \bigcup_{a \in B} \theta\pi(a)$. On the other hand, for any $c \in \bigcup_{a \in B} \theta\pi(a)$, there exists $a_0 \in B$ such that $c \in \theta\pi(a_0)$, i.e., $\pi(a_0) \subseteq \theta(c)$. Since $X \subseteq \pi(B) \subseteq \pi(a_0)$, it follows $X \subseteq \theta(c)$. Note that $c \in \theta\theta(c) \subseteq \theta(X) = C$. Then, we conclude $\bigcup_{a \in B} \theta\pi(a) \subseteq C$. In summary, $C = \bigcup_{a \in B} \theta\pi(a)$ is proved.

According to Theorem 2, we obtain $X = g(B, C) = \bigcap_{a \in B} g^G f\pi(a)$. Then, we know from Eq. (7) that $(X, (B, C))$ and $\bigwedge_{a \in B} (g^G f\pi(a), f\pi(a))$ have the same extent. As a result, they are the same approximate cognitive concept. Consequently, (ii) is true. $\square$

Theorem 4 indicates that any approximate cognitive concept can be induced by granular concepts. So, among all approximate cognitive concepts, granular concepts are the basic information granules. For convenience, we denote the set of all granular concepts by

$$G_{fg} = \left\{ (gf^G(x), f^G(x)) \middle| x \in U \right\} \cup \left\{ (g^G f\pi(a), f\pi(a)) \middle| a \in A \right\}. \tag{12}$$

In the rest of this paper, we discuss how to compute $G_{fg}$ when objects and attributes are updated in batch.

## 3.2 A cognitive computing system for learning granular concepts

To facilitate our subsequent discussions, $n$ object sets $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n$ are denoted as $\{U_t\}^{\uparrow}$, and $n$ attribute sets $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$ are denoted as $\{A_t\}^{\uparrow}$. Let $\Delta U_{i-1} = U_i - U_{i-1}$ and $\Delta A_{i-1} = A_i - A_{i-1}$.

To express and process different types of incomplete information, let $I_{i-1}$ be a "three-valued" mapping from $U_{i-1} \times A_{i-1}$ to $\{1, 0, ?\}$, $I_{\Delta U_{i-1}}$ be a "three-valued" mapping from $\Delta U_{i-1} \times A_{i-1}$ to $\{1, 0, ?\}$, $I_{\Delta A_{i-1}}$ be a "three-valued" mapping from $U_i \times \Delta A_{i-1}$ to $\{1, 0, ?\}$, and $I_i$ be a "three-valued" mapping from $U_i \times A_i$ to $\{1, 0, ?\}$.

Then, we define sixteen mappings

$$\pi_{i-1}(X) = \{a \in A_{i-1} | \forall x \in X, I_{i-1}(x, a) = 1\},$$
$$\theta_{i-1}(X) = \{a \in A_{i-1} | \forall x \in X, I_{i-1}(x, a) = 1 \text{ or } I_{i-1}(x, a) = ?\},$$
$$\pi_{i-1}(B) = \{x \in U_{i-1} | \forall a \in B, I_{i-1}(x, a) = 1\},$$
$$\theta_{i-1}(B) = \{x \in U_{i-1} | \forall a \in B, I_{i-1}(x, a) = 1 \text{ or } I_{i-1}(x, a) = ?\},$$

$$\pi_{\Delta U_{i-1}}(X) = \{a \in A_{i-1} | \forall x \in X, I_{\Delta U_{i-1}}(x, a) = 1\},$$
$$\theta_{\Delta U_{i-1}}(X) = \{a \in A_{i-1} | \forall x \in X, I_{\Delta U_{i-1}}(x, a) = 1 \text{ or } I_{\Delta U_{i-1}}(x, a) = ?\},$$
$$\pi_{\Delta U_{i-1}}(B) = \{x \in \Delta U_{i-1} | \forall a \in B, I_{\Delta U_{i-1}}(x, a) = 1\},$$
$$\theta_{\Delta U_{i-1}}(B) = \{x \in \Delta U_{i-1} | \forall a \in B, I_{\Delta U_{i-1}}(x, a) = 1 \text{ or } I_{\Delta U_{i-1}}(x, a) = ?\},$$

$$\pi_{\Delta A_{i-1}}(X) = \{a \in \Delta A_{i-1} | \forall x \in X, I_{\Delta A_{i-1}}(x, a) = 1\},$$
$$\theta_{\Delta A_{i-1}}(X) = \{a \in \Delta A_{i-1} | \forall x \in X, I_{\Delta A_{i-1}}(x, a) = 1 \text{ or } I_{\Delta A_{i-1}}(x, a) = ?\},$$
$$\pi_{\Delta A_{i-1}}(B) = \{x \in U_i | \forall a \in B, I_{\Delta A_{i-1}}(x, a) = 1\},$$
$$\theta_{\Delta A_{i-1}}(B) = \{x \in U_i | \forall a \in B, I_{\Delta A_{i-1}}(x, a) = 1 \text{ or } I_{\Delta A_{i-1}}(x, a) = ?\},$$

$$\pi_i(X) = \{a \in A_i | \forall x \in X, I_i(x, a) = 1\},$$
$$\theta_i(X) = \{a \in A_i | \forall x \in X, I_i(x, a) = 1 \text{ or } I_i(x, a) = ?\},$$
$$\pi_i(B) = \{x \in U_i | \forall a \in B, I_i(x, a) = 1\},$$
$$\theta_i(B) = \{x \in U_i | \forall a \in B, I_i(x, a) = 1 \text{ or } I_i(x, a) = ?\}.$$

Furthermore, four pairs of approximate cognitive operators are defined as

(i) $f_{i-1} : 2^{U_{i-1}} \to 2^{A_{i-1}} \times 2^{A_{i-1}}, g_{i-1} : 2^{A_{i-1}} \times 2^{A_{i-1}} \to 2^{U_{i-1}}$,

(ii) $f_{\Delta U_{i-1}} : 2^{\Delta U_{i-1}} \to 2^{A_{i-1}} \times 2^{A_{i-1}}, g_{\Delta U_{i-1}} : 2^{A_{i-1}} \times 2^{A_{i-1}} \to 2^{\Delta U_{i-1}}$,

(iii) $f_{\Delta A_{i-1}} : 2^{U_i} \to 2^{\Delta A_{i-1}} \times 2^{\Delta A_{i-1}}, g_{\Delta A_{i-1}} : 2^{\Delta A_{i-1}} \times 2^{\Delta A_{i-1}} \to 2^{U_i}$,

(iv) $f_i : 2^{U_i} \to 2^{A_i} \times 2^{A_i}, g_i : 2^{A_i} \times 2^{A_i} \to 2^{U_i}$,

where

$$f_{i-1}(X) = (\pi_{i-1}(X), \theta_{i-1}(X)),$$
$$g_{i-1}(B, C) = \{x \in U_{i-1} | (B, C) \leq f_{i-1}(x)\}, \quad (13)$$

$$f_{\Delta U_{i-1}}(X) = (\pi_{\Delta U_{i-1}}(X), \theta_{\Delta U_{i-1}}(X)),$$
$$g_{\Delta U_{i-1}}(B, C) = \{x \in \Delta U_{i-1} | (B, C) \leq f_{\Delta U_{i-1}}(x)\}, \quad (14)$$

$$f_{\Delta A_{i-1}}(X) = (\pi_{\Delta A_{i-1}}(X), \theta_{\Delta A_{i-1}}(X)),$$
$$g_{\Delta A_{i-1}}(B, C) = \{x \in U_i | (B, C) \leq f_{\Delta A_{i-1}}(x)\}, \quad (15)$$

$$f_i(X) = (\pi_i(X), \theta_i(X)),$$
$$g_i(B, C) = \{x \in U_i | (B, C) \leq f_i(x)\}. \quad (16)$$

In fact, the above approximate cognitive operators: (i) $f_{i-1}, g_{i-1}$, (ii) $f_{\Delta U_{i-1}}, g_{\Delta U_{i-1}}$, (iii) $f_{\Delta A_{i-1}}, g_{\Delta A_{i-1}}$ and (iv) $f_i, g_i$ are formed by adding objects and attributes in sequence. In order to establish a cognitive computing system for updating granular concepts, we need to further clarify their inner relationships.

**Theorem 5** *Let* (i) $f_{i-1}, g_{i-1}$, (ii) $f_{\Delta U_{i-1}}, g_{\Delta U_{i-1}}$, (iii) $f_{\Delta A_{i-1}}, g_{\Delta A_{i-1}}$ *and* (iv) $f_i, g_i$ *be the approximate cognitive operators specified in Eqs.* (13)–(16). *Then, we have*

$$f_i^G(x) = \begin{cases} f_{i-1}^G(x) \cup f_{\Delta A_{i-1}}^G(x), & \text{if } x \in U_{i-1}, \\ f_{\Delta U_{i-1}}^G(x) \cup f_{\Delta A_{i-1}}^G(x), & \text{otherwise.} \end{cases} \quad (17)$$

*Proof* By Eq. (16), it follows $f_i^G(x) = (\pi_i(x), \theta_i(x))$. If $x \in U_{i-1}$, we obtain

$$\begin{aligned} \pi_i(x) &= \{a \in A_i | I_i(x, a) = 1\} \\ &= \{a \in A_{i-1} \cup \Delta A_{i-1} | I_i(x, a) = 1\} \\ &= \{a \in A_{i-1} | I_i(x, a) = 1\} \cup \{a \in \Delta A_{i-1} | I_i(x, a) = 1\} \\ &= \pi_{i-1}(x) \cup \pi_{\Delta A_{i-1}}(x), \end{aligned}$$

$$\begin{aligned} \theta_i(x) &= \{a \in A_i | I_i(x, a) = 1 \text{ or } I_i(x, a) = ?\} \\ &= \{a \in A_{i-1} \cup \Delta A_{i-1} | I_i(x, a) = 1 \text{ or } I_i(x, a) = ?\} \\ &= \{a \in A_{i-1} | I_i(x, a) = 1 \text{ or } I_i(x, a) = ?\} \\ &\cup \{a \in \Delta A_{i-1} | I_i(x, a) = 1 \text{ or } I_i(x, a) = ?\} \\ &= \theta_{i-1}(x) \cup \theta_{\Delta A_{i-1}}(x). \end{aligned}$$

Therefore, we get

$$\begin{aligned} f_i^G(x) &= (\pi_i(x), \theta_i(x)) \\ &= (\pi_{i-1}(x) \cup \pi_{\Delta A_{i-1}}(x), \theta_{i-1}(x) \cup \theta_{\Delta A_{i-1}}(x)) \\ &= (\pi_{i-1}(x), \theta_{i-1}(x)) \cup (\pi_{\Delta A_{i-1}}(x), \theta_{\Delta A_{i-1}}(x)) \\ &= f_{i-1}^G(x) \cup f_{\Delta A_{i-1}}^G(x). \end{aligned}$$

The conclusion $f_i^G(x) = f_{\Delta U_{i-1}}^G(x) \cup f_{\Delta A_{i-1}}^G(x)$ can be proved in a similar manner when $x \in \Delta U_{i-1}$. $\qquad \square$

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Theorem 5 shows how to obtain $f_i^G(x)$ ($x \in U_i$). In fact, they can further be used to compute

$$g_i f_i^G(x) = \left\{ y \in U_i | f_i^G(x) \leq f_i^G(y) \right\}. \tag{18}$$

That is, we find an approach to generate the granular concepts $\left( g_i f_i^G(x), f_i^G(x) \right)$. At the same time, $f_i^G(x)$ ($x \in U_i$) can be used to compute

$$f_i \pi_i(a) = \bigcap_{x \in \pi_i(a)} f_i^G(x) \tag{19}$$

and

$$g_i^G f_i \pi_i(a) = \left\{ x \in U_i | f_i \pi_i(a) \leq f_i^G(x) \right\}. \tag{20}$$

That is to say, we also find an approach to generate the granular concepts $\left( g_i^G f_i \pi_i(a), f_i \pi_i(a) \right)$ ($a \in A_i$). To sum up, we have the following theorem.

**Theorem 6** *Let* (i) $f_{i-1}, g_{i-1}$, (ii) $f_{\Delta U_{i-1}}, g_{\Delta U_{i-1}}$, (iii) $f_{\Delta A_{i-1}}, g_{\Delta A_{i-1}}$ *and* (iv) $f_i, g_i$ *be the approximate cognitive operators specified in Eqs.* (13)–(16). *Then, the granular concepts* $G_{f_i g_i}$ *of* $f_i$ *and* $g_i$ *can be computed by combining* $G_{f_{i-1} g_{i-1}}$ *with* $f_{\Delta U_{i-1}}^G$ *and* $f_{\Delta A_{i-1}}^G$.

*Proof* It is immediate from Eqs. (17), (18), (19) and (20). □

**Definition 5** *Let* (i) $f_{i-1}, g_{i-1}$, (ii) $f_{\Delta U_{i-1}}, g_{\Delta U_{i-1}}$, (iii) $f_{\Delta A_{i-1}}, g_{\Delta A_{i-1}}$ *and* (iv) $f_i, g_i$ *be the approximate cognitive operators specified in Eqs.* (13)–(16). *We say that* $\mathcal{AS}_{f_i g_i} = (G_{f_{i-1} g_{i-1}}, f_{\Delta U_{i-1}}^G, f_{\Delta A_{i-1}}^G)$ *is an approximate cognitive computing state. Furthermore,* $\mathcal{AS} = \bigcup_{i=2}^{n} \{ \mathcal{AS}_{f_i g_i} \}$ *is said to be an approximate cognitive computing system.*

It can be known from Definition 5 that an approximate cognitive computing system is constituted by a series of approximate cognitive computing states. Notice that granular concepts with objects and attributes being updated once are considered as a state. This is in accordance with our common sense. In fact, it is impossible to directly obtain the granular concepts of an approximate cognitive computing system since the information is updated dynamically. In other words, we only know the granular concepts from one state to another. Here, we adopt the recursive strategy to compute the granular concepts $G_{f_1 g_1}$, $G_{f_2 g_2}$, $\cdots$, $G_{f_n g_n}$ in sequence.

Based on the above discussion, we are ready to propose a procedure (called Algorithm 1) to compute the granular concepts $G_{f_n g_n}$ of an approximate cognitive computing system $\mathcal{AS} = \bigcup_{i=2}^{n} \{ \mathcal{AS}_{f_i g_i} \}$.

---

**Algorithm 1** Computing the granular concepts of an approximate cognitive computing system

*Input*: $\mathcal{AS} = \bigcup_{i=2}^{n} \{ \mathcal{AS}_{\mathcal{L}_i \mathcal{H}_i} \}$, where $\mathcal{AS}_{\mathcal{L}_i \mathcal{H}_i} = (G_{f_{i-1} g_{i-1}}, f_{\Delta U_{i-1}}^G, f_{\Delta A_{i-1}}^G)$ is an approximate cognitive computing state.

*Output*: The granular concepts $G_{f_n g_n}$ of $\mathcal{AS}$.

1. Initialize $G_{f_1 g_1} = \{ (g_1 f_1^G(x), f_1^G(x)) \mid x \in U_1 \} \cup \{ (g_1^G f_1 \pi_1(a), f_1 \pi_1(a)) \mid a \in A_1 \}$ and $i = 2$;
2. **While** $i \leq n$
3.   Denote the $i$-th approximate cognitive computing state by $\mathcal{AS}_{\mathcal{L}_i \mathcal{H}_i} = (G_{f_{i-1} g_{i-1}}, f_{\Delta U_{i-1}}^G, f_{\Delta A_{i-1}}^G)$;
4.     **For** each $x \in U_i$
5.       **If** $x \in U_{i-1}$
6.         set $f_i^G(x) = f_{i-1}^G(x) \cup f_{\Delta A_{i-1}}^G(x)$;
7.       **Else**
8.         set $f_i^G(x) = f_{\Delta U_{i-1}}^G(x) \cup f_{\Delta A_{i-1}}^G(x)$;
9.       **End If**
10.    **End For**
11.    **For** each $x \in U_i$
12.      compute $g_i f_i^G(x) = \{ y \in U_i | f_i^G(x) \leq f_i^G(y) \}$;
13.    **End For**
14.    **For** each $a \in A_i$
15.      compute $f_i \pi_i(a) = \bigcap_{x \in \pi_i(a)} f_i^G(x)$ and

$$g_i^G f_i \pi_i(a) = \{ x \in U_i | \bigcap_{y \in \pi_i(a)} f_i^G(y) \leq f_i^G(x) \};$$

16.    **End For**
17.    Set $G_{f_i g_i} = \left\{ (g_i f_i^G(x), f_i^G(x)) \mid x \in U_i \right\} \cup \{ (g_i^G f_i \pi_i(a), f_i \pi_i(a)) \mid a \in A_i \}$;
18.    $i \leftarrow i + 1$;
19. **End While**
20. Output $G_{f_i g_i}$ and end the algorithm.

---

The correctness of Algorithm 1 is guaranteed by Eqs. (17), (18), (19) and (20). Its time complexity is analyzed as follows. Based on Eq. (12), running Step 1 takes $O((|U_1| + |A_1|)|U_1||A_1|)$. Similarly, running Steps 3–18 takes $O((|U_i| + |A_i|)|U_i||A_i|)$. So, the time complexity of Algorithm 1 is $O(n(|U_n| + |A_n|)|U_n||A_n|)$, where $n$ is the number of approximate cognitive computing states. That is, Algorithm 1 needs polynomial time.

Finally, we use an example to illustrate Algorithm 1.

*Example 3* Continued with Example 2. Let $U_1 = \{1, 2, 3, 4\}$ and $A_1 = \{a, b, c, d\}$. Then, we have the following equalities:

$f_1^G(1) = (abd, abd), f_1^G(2) = (bd, bd), f_1^G(3) = (c, c), f_1^G(4) = (ac, ac)$,

$g_1 f_1^G(1) = \{1\}, g_1 f_1^G(2) = \{1, 2\}, g_1 f_1^G(3) = \{3, 4\}, g_1 f_1^G(4) = \{4\}$,

$\pi_1(a) = \{1, 4\}, \pi_1(b) = \{1, 2\}, \pi_1(c) = \{3, 4\}, \pi_1(d) = \{1, 2\}$,

$f_1 \pi_1(a) = (a, a), f_1 \pi_1(b) = (bd, bd), f_1 \pi_1(c) = (c, c), f_1 \pi_1(d) = (bd, bd)$,

$g_1^G f_1 \pi_1(a) = \{1, 4\}, g_1^G f_1 \pi_1(b) = \{1, 2\}$,

$g_1^G f_1 \pi_1(c) = \{3, 4\}, g_1^G f_1 \pi_1(d) = \{1, 2\}$.

Based on Eq. (12), the granular concepts of $f_1$ and $g_1$ are as follows:

$$G_{f_1 g_1} = \{ (1, (abd, abd)), (4, (ac, ac)), (12, (bd, bd)),$$
$$(14, (a, a)), (34, (c, c)) \}.$$

Take $\Delta U_1 = \{5, 6, 7, 8, 9\}$. We have

$f_{\Delta U_1}^G(5) = (ac, ac), f_{\Delta U_1}^G(6) = (b, b), f_{\Delta U_1}^G(7) = (ad, ad),$
$f_{\Delta U_1}^G(8) = (b, b), f_{\Delta U_1}^G(9) = (c, c).$

Set $\Delta A_1 = \{e, f, g\}$. It follows

$f_{\Delta A_1}^G(1) = (\emptyset, efg), f_{\Delta A_1}^G(2) = (\emptyset, efg), f_{\Delta A_1}^G(3) = (\emptyset, efg),$
$f_{\Delta A_1}^G(4) = (\emptyset, efg), f_{\Delta A_1}^G(5) = (e, e), f_{\Delta A_1}^G(6) = (f, f),$
$f_{\Delta A_1}^G(7) = (g, g), f_{\Delta A_1}^G(8) = (fg, fg), f_{\Delta A_1}^G(9) = (e, e).$

Furthermore, let $U_2 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A_2 = \{a, b, c, d, e, f, g\}$. By Eq. (17), we obtain

$f_2^G(1) = f_1^G(1) \cup f_{\Delta A_1}^G(1) = (abd, abd) \cup (\emptyset, efg) = (abd, abdefg),$
$f_2^G(2) = f_1^G(2) \cup f_{\Delta A_1}^G(2) = (bd, bd) \cup (\emptyset, efg) = (bd, bdefg),$
$f_2^G(3) = f_1^G(3) \cup f_{\Delta A_1}^G(3) = (c, c) \cup (\emptyset, efg) = (c, cefg),$
$f_2^G(4) = f_1^G(4) \cup f_{\Delta A_1}^G(4) = (ac, ac) \cup (\emptyset, efg) = (ac, acefg),$
$f_2^G(5) = f_{\Delta U_1}^G(5) \cup f_{\Delta A_1}^G(5) = (ac, ac) \cup (e, e) = (ace, ace),$
$f_2^G(6) = f_{\Delta U_1}^G(6) \cup f_{\Delta A_1}^G(6) = (b, b) \cup (f, f) = (bf, bf),$
$f_2^G(7) = f_{\Delta U_1}^G(7) \cup f_{\Delta A_1}^G(7) = (ad, ad) \cup (g, g) = (adg, adg),$
$f_2^G(8) = f_{\Delta U_1}^G(8) \cup f_{\Delta A_1}^G(8) = (b, b) \cup (fg, fg) = (bfg, bfg),$
$f_2^G(9) = f_{\Delta U_1}^G(9) \cup f_{\Delta A_1}^G(9) = (c, c) \cup (e, e) = (ce, ce).$

According to Eq. (18), we get

$g_2 f_2^G(1) = \{x \in U_2 | f_2^G(1) \leq f_2^G(x)\} = \{1\},$
$g_2 f_2^G(2) = \{x \in U_2 | f_2^G(2) \leq f_2^G(x)\} = \{1, 2\},$
$g_2 f_2^G(3) = \{x \in U_2 | f_2^G(3) \leq f_2^G(x)\} = \{3, 4\},$
$g_2 f_2^G(4) = \{x \in U_2 | f_2^G(4) \leq f_2^G(x)\} = \{4\},$
$g_2 f_2^G(5) = \{x \in U_2 | f_2^G(5) \leq f_2^G(x)\} = \{5\},$
$g_2 f_2^G(6) = \{x \in U_2 | f_2^G(6) \leq f_2^G(x)\} = \{6, 8\},$
$g_2 f_2^G(7) = \{x \in U_2 | f_2^G(7) \leq f_2^G(x)\} = \{7\},$
$g_2 f_2^G(8) = \{x \in U_2 | f_2^G(8) \leq f_2^G(x)\} = \{8\},$
$g_2 f_2^G(9) = \{x \in U_2 | f_2^G(9) \leq f_2^G(x)\} = \{5, 9\}.$

Based on Eq. (19), we conclude

$\pi_2(a) = \{1, 4, 5, 7\}, f_2 \pi_2(a) = \bigcap_{x \in \pi_2(a)} f_2^G(x) = (a, a),$
$\pi_2(b) = \{1, 2, 6, 8\}, f_2 \pi_2(b) = \bigcap_{x \in \pi_2(b)} f_2^G(x) = (b, bf),$
$\pi_2(c) = \{3, 4, 5, 9\}, f_2 \pi_2(c) = \bigcap_{x \in \pi_2(c)} f_2^G(x) = (c, ce),$
$\pi_2(d) = \{1, 2, 7\}, f_2 \pi_2(d) = \bigcap_{x \in \pi_2(d)} f_2^G(x) = (d, dg),$
$\pi_2(e) = \{5, 9\}, f_2 \pi_2(e) = \bigcap_{x \in \pi_2(e)} f_2^G(x) = (ce, ce),$
$\pi_2(f) = \{6, 8\}, f_2 \pi_2(f) = \bigcap_{x \in \pi_2(f)} f_2^G(x) = (bf, bf),$
$\pi_2(g) = \{7, 8\}, f_2 \pi_2(g) = \bigcap_{x \in \pi_2(g)} f_2^G(x) = (g, g).$

Then, it follows from Eq. (20) that

$g_2 f_2 \pi_2(a) = \{x \in U_2 | f_2 \pi_2(a) \leq f_2^G(x)\} = \{1, 4, 5, 7\},$
$g_2 f_2 \pi_2(b) = \{x \in U_2 | f_2 \pi_2(b) \leq f_2^G(x)\} = \{1, 2, 6, 8\},$
$g_2 f_2 \pi_2(c) = \{x \in U_2 | f_2 \pi_2(c) \leq f_2^G(x)\} = \{3, 4, 5, 9\},$
$g_2 f_2 \pi_2(d) = \{x \in U_2 | f_2 \pi_2(d) \leq f_2^G(x)\} = \{1, 2, 7\},$
$g_2 f_2 \pi_2(e) = \{x \in U_2 | f_2 \pi_2(e) \leq f_2^G(x)\} = \{5, 9\},$
$g_2 f_2 \pi_2(f) = \{x \in U_2 | f_2 \pi_2(f) \leq f_2^G(x)\} = \{6, 8\},$
$g_2 f_2 \pi_2(g) = \{x \in U_2 | f_2 \pi_2(g) \leq f_2^G(x)\} = \{7, 8\}.$

Consequently, the granular concepts of the approximate cognitive computing system $\mathcal{AS} = \mathcal{AS}_{f_2 g_2}$ are

$(X_1, (B_1, C_1)) = (1, (abd, abdefg)), (X_2, (B_2, C_2)) = (12, (bd, bdefg)),$
$(X_3, (B_3, C_3)) = (34, (c, cefg)), (X_4, (B_4, C_4)) = (4, (ac, acefg)),$
$(X_5, (B_5, C_5)) = (5, (ace, ace)), (X_6, (B_6, C_6)) = (68, (bf, bf)),$
$(X_7, (B_7, C_7)) = (7, (adg, adg)), (X_8, (B_8, C_8)) = (8, (bfg, bfg)),$
$(X_9, (B_9, C_9)) = (59, (ce, ce)), (X_{10}, (B_{10}, C_{10})) = (1457, (a, a)),$
$(X_{11}, (B_{11}, C_{11})) = (1268, (b, bf)), (X_{12}, (B_{12}, C_{12})) = (3459, (c, ce)),$
$(X_{13}, (B_{13}, C_{13})) = (127, (d, dg)), (X_{14}, (B_{14}, C_{14})) = (78, (g, g)).$

## 4 Cognitive processes

In Sect. 3, granular concepts were generated by an approximate cognitive computing system. From the viewpoint of cognitive computing, granular concepts are the basic information granules stored in human brain. These granular concepts will be reactivated when a clue comes into mind, which is often called a cognitive process.

Since a clue may be objects, attributes or both of them, cognitive processes will be investigated in this section from three aspects: a) learn granular concept from an object set; b) learn granular concept from a pair of attribute sets; c) learn granular concept from both of them.

### 4.1 Granular concept learning from an object set based on similarity

In cognitive computing, similarity is commonly adopted to remember something when a clue is provided. Up to now, there have been many methods for measuring concept similarity. More details are omitted here. Interested readers can refer to [6, 35].

The clue is assumed in this subsection to be a nonempty object set $X_0$. According to the above discussion, the granular concepts $G_{f_n g_n}$ of an approximate cognitive computing system $\mathcal{AS} = \bigcup_{i=2}^n \{\mathcal{AS}_{\mathcal{L}_i \mathcal{H}_i}\}$ are the basic knowledge used to achieve the learning task.

In what follows, we first introduce the notion of similarity between two nonempty object sets.

**Definition 6** The similarity between two nonempty object sets $X_1, X_2 \subseteq U_n$ is defined as

$$s_u(X_1, X_2) = \frac{1}{2}\left(\frac{|X_1 \cap X_2|}{|X_1|} + \frac{|X_2 \cap X_1|}{|X_2|}\right). \tag{21}$$

According to Definition 6, we have the following proposition.

**Proposition 1** *For nonempty object sets $X, X_1, X_2 \subseteq U_n$, both $s_u(X, X) = 1$ and $s_u(X_1, X_2) = s_u(X_2, X_1)$ are true.*

Then, based on this similarity, granular concept learning from a nonempty object set $X_0$ is to select the granular concept $(X, (B, C))$ which satisfies

$$s_u(X, X_0) = \max_{(X',(B',C'))\in G_{f_n g_n}} \{s_u(X', X_0)\}. \tag{22}$$

*Example 4* Continued with Example 3. Suppose that the given clue is $X_0 = \{4, 5, 9\}$. Then, based on Eq. (21), it follows

$$s_u(X_1, X_0) = \frac{1}{2}\left(\frac{|X_1 \cap X_0|}{|X_1|} + \frac{|X_0 \cap X_1|}{|X_0|}\right)$$
$$= \frac{1}{2}\left(\frac{|\{1\} \cap \{4,5,9\}|}{|\{1\}|} + \frac{|\{4,5,9\} \cap \{1\}|}{|\{4,5,9\}|}\right) = 0.$$

Similarly, we obtain the following equalities:

$$s_u(X_2, X_0) = \frac{1}{2}\left(\frac{|X_2 \cap X_0|}{|X_2|} + \frac{|X_0 \cap X_2|}{|X_0|}\right) = 0,$$
$$s_u(X_3, X_0) = \frac{1}{2}\left(\frac{|X_3 \cap X_0|}{|X_3|} + \frac{|X_0 \cap X_3|}{|X_0|}\right) = \frac{5}{12},$$
$$s_u(X_4, X_0) = \frac{1}{2}\left(\frac{|X_4 \cap X_0|}{|X_4|} + \frac{|X_0 \cap X_4|}{|X_0|}\right) = \frac{2}{3},$$
$$s_u(X_5, X_0) = \frac{1}{2}\left(\frac{|X_5 \cap X_0|}{|X_5|} + \frac{|X_0 \cap X_5|}{|X_0|}\right) = \frac{2}{3},$$
$$s_u(X_6, X_0) = \frac{1}{2}\left(\frac{|X_6 \cap X_0|}{|X_6|} + \frac{|X_0 \cap X_6|}{|X_0|}\right) = 0,$$
$$s_u(X_7, X_0) = \frac{1}{2}\left(\frac{|X_7 \cap X_0|}{|X_7|} + \frac{|X_0 \cap X_7|}{|X_0|}\right) = 0,$$
$$s_u(X_8, X_0) = \frac{1}{2}\left(\frac{|X_8 \cap X_0|}{|X_8|} + \frac{|X_0 \cap X_8|}{|X_0|}\right) = 0,$$
$$s_u(X_9, X_0) = \frac{1}{2}\left(\frac{|X_9 \cap X_0|}{|X_9|} + \frac{|X_0 \cap X_9|}{|X_0|}\right) = \frac{5}{6},$$
$$s_u(X_{10}, X_0) = \frac{1}{2}\left(\frac{|X_{10} \cap X_0|}{|X_{10}|} + \frac{|X_0 \cap X_{10}|}{|X_0|}\right) = \frac{7}{12},$$
$$s_u(X_{11}, X_0) = \frac{1}{2}\left(\frac{|X_{11} \cap X_0|}{|X_{11}|} + \frac{|X_0 \cap X_{11}|}{|X_0|}\right) = 0,$$
$$s_u(X_{12}, X_0) = \frac{1}{2}\left(\frac{|X_{12} \cap X_0|}{|X_{12}|} + \frac{|X_0 \cap X_{12}|}{|X_0|}\right) = \frac{7}{8},$$
$$s_u(X_{13}, X_0) = \frac{1}{2}\left(\frac{|X_{13} \cap X_0|}{|X_{13}|} + \frac{|X_0 \cap X_{13}|}{|X_0|}\right) = 0,$$
$$s_u(X_{14}, X_0) = \frac{1}{2}\left(\frac{|X_{14} \cap X_0|}{|X_{14}|} + \frac{|X_0 \cap X_{14}|}{|X_0|}\right) = 0.$$

That is, $(X_{12}, (B_{12}, C_{12})) = (3459, (c, ce))$ with the similarity $s_u(X_{12}, X_0) = \frac{7}{8}$ is the best granular concept which matches the given clue.

## 4.2 Granular concept learning from a pair of attribute sets based on similarity

Similarly, to learn granular concept from a pair of attribute sets, similarity between attributes sets is also required.

**Definition 7** Let $(B_1, C_1), (B_2, C_2) \in 2^{A_n} \times 2^{A_n}$ and $\lambda \in (0, 1)$. Then, the similarity between $(B_1, C_1)$ and $(B_2, C_2)$ is defined as

$$s_{aa}((B_1, C_1), (B_2, C_2)) = \lambda s_a(B_1, B_2) + (1 - \lambda)s_a(C_1, C_2), \tag{23}$$

where $s_a(\cdot, \cdot)$ is the similarity similar to the one specified in Eq. (21).

Note that in Definition 7, $\lambda$ is a weight parameter used to adjust the significance of the certain and possible parts. Furthermore, we have the following proposition.

**Proposition 2** *For $(B, C), (B_1, C_1), (B_2, C_2) \in 2^{A_n} \times 2^{A_n}$, we have $s_{aa}((B, C), (B, C)) = 1$ and $s_{aa}((B_1, C_1), (B_2, C_2)) = s_{aa}((B_2, C_2), (B_1, C_1))$.*

Then, based on this similarity, granular concept learning from a pair of attribute sets $(B_0, C_0)$ is to select the granular concept $(X, (B, C))$ which satisfies

$$s_{aa}((B, C), (B_0, C_0))$$
$$= \max_{(X',(B',C'))\in G_{f_n g_n}} \{s_{aa}((B', C'), (B_0, C_0))\}. \tag{24}$$

*Example 5* Continued with Example 3. Suppose that the given clue is $(B_0, C_0) = (ac, ce)$ and $\lambda = \frac{3}{5}$. Then, based on Eq. (23), it follows

$$s_{aa}((B_1, C_1), (B_0, C_0)) = \lambda s_a(B_1, B_0) + (1 - \lambda)s_a(C_1, C_0)$$
$$= \frac{\lambda}{2}\left(\frac{|B_1 \cap B_0|}{|B_1|} + \frac{|B_0 \cap B_1|}{|B_0|}\right)$$
$$+ \frac{1 - \lambda}{2}\left(\frac{|C_1 \cap C_0|}{|C_1|} + \frac{|C_0 \cap C_1|}{|C_0|}\right)$$
$$= \frac{23}{60}.$$

Similarly, we obtain the following equalities:

$$s_{aa}((B_2, C_2), (B_0, C_0)) = \lambda s_a(B_2, B_0) + (1 - \lambda)s_a(C_2, C_0) = \frac{7}{50},$$
$$s_{aa}((B_3, C_3), (B_0, C_0)) = \lambda s_a(B_3, B_0) + (1 - \lambda)s_a(C_3, C_0) = \frac{3}{4},$$
$$s_{aa}((B_4, C_4), (B_0, C_0)) = \lambda s_a(B_4, B_0) + (1 - \lambda)s_a(C_4, C_0) = \frac{22}{25},$$
$$s_{aa}((B_5, C_5), (B_0, C_0)) = \lambda s_a(B_5, B_0) + (1 - \lambda)s_a(C_5, C_0) = \frac{5}{6},$$
$$s_{aa}((B_6, C_6), (B_0, C_0)) = \lambda s_a(B_6, B_0) + (1 - \lambda)s_a(C_6, C_0) = 0,$$
$$s_{aa}((B_7, C_7), (B_0, C_0)) = \lambda s_a(B_7, B_0) + (1 - \lambda)s_a(C_7, C_0) = \frac{1}{4},$$
$$s_{aa}((B_8, C_8), (B_0, C_0)) = \lambda s_a(B_8, B_0) + (1 - \lambda)s_a(C_8, C_0) = 0,$$

$$s_{aa}((B_9, C_9), (B_0, C_0)) = \lambda s_a(B_9, B_0) + (1 - \lambda)s_a(C_9, C_0) = \frac{7}{10},$$

$$s_{aa}((B_{10}, C_{10}), (B_0, C_0)) = \lambda s_a(B_{10}, B_0) + (1 - \lambda)s_a(C_{10}, C_0) = \frac{9}{20},$$

$$s_{aa}((B_{11}, C_{11}), (B_0, C_0)) = \lambda s_a(B_{11}, B_0) + (1 - \lambda)s_a(C_{11}, C_0) = 0,$$

$$s_{aa}((B_{12}, C_{12}), (B_0, C_0)) = \lambda s_a(B_{12}, B_0) + (1 - \lambda)s_a(C_{12}, C_0) = \frac{51}{60},$$

$$s_{aa}((B_{13}, C_{13}), (B_0, C_0)) = \lambda s_a(B_{13}, B_0) + (1 - \lambda)s_a(C_{13}, C_0) = 0,$$

$$s_{aa}((B_{14}, C_{14}), (B_0, C_0)) = \lambda s_a(B_{14}, B_0) + (1 - \lambda)s_a(C_{14}, C_0) = 0.$$

That is, $(X_{12}, (B_{12}, C_{12})) = (3459, (c, ce))$ with the similarity $s_{aa}((B_{12}, C_{12}), (B_0, C_0)) = \frac{51}{60}$ is the best granular concept which matches the given clue.

### 4.3 Granular concept learning from an object set and a pair of attribute sets based on similarity

**Definition 8** Let $X_1, X_2 \subseteq U_n$, $(B_1, C_1), (B_2, C_2) \in 2^{A_n} \times 2^{A_n}$ and $\mu \in (0, 1)$. Then, the similarity between $(X_1, (B_1, C_1))$ and $(X_2, (B_2, C_2))$ is defined as

$$s((X_1, (B_1, C_1)), (X_2, (B_2, C_2)))$$
$$= \mu s_u(X_1, X_2) + (1 - \mu)s_{aa}((B_1, C_1), (B_2, C_2)).$$

Note that in Definition 8, $\mu$ is a weight parameter used to adjust the significance of an object set and a pair of attribute sets. Furthermore, we have the following proposition.

**Proposition 3** Let $X, X_1, X_2 \subseteq U_n$ and $(B, C), (B_1, C_1), (B_2, C_2) \in 2^{A_n} \times 2^{A_n}$. Then, $s((X, (B, C)), (X, (B, C))) = 1$ and $s((X_1, (B_1, C_1)), (X_2, (B_2, C_2))) = s((X_2, (B_2, C_2)), (X_1, (B_1, C_1)))$.

Then, based on this similarity, granular concept learning from an object set $X_0$ and a pair of attribute sets $(B_0, C_0)$ is to select the granular concept $(X, (B, C))$ which satisfies

$$s((X, (B, C)), (X_0, (B_0, C_0)))$$
$$= \max_{(X', (B', C')) \in G_{f_n g_n}} \{s((X', (B', C')), (X_0, (B_0, C_0)))\}.$$

*Example 6* Continued with Example 3. Suppose that the given clue is constituted by $X_0 = \{4, 5, 9\}$ and $(B_0, C_0) = (ac, ce)$. Take $\mu = \lambda = \frac{3}{5}$. Then, it follows from Definition 8 that

$$s((X_1, (B_1, C_1)), (X_0, (B_0, C_0)))$$
$$= \mu s_u(X_1, X_0) + (1 - \mu)s_{aa}((B_1, C_1), (B_0, C_0)) = \frac{23}{60}.$$

Similarly, we obtain the following equalities:

$$s((X_2, (B_2, C_2)), (X_0, (B_0, C_0))) = \frac{21}{250},$$

$$s((X_3, (B_3, C_3)), (X_0, (B_0, C_0))) = \frac{37}{60},$$

$$s((X_4, (B_4, C_4)), (X_0, (B_0, C_0))) = \frac{298}{375},$$

$$s((X_5, (B_5, C_5)), (X_0, (B_0, C_0))) = \frac{23}{30},$$

$$s((X_6, (B_6, C_6)), (X_0, (B_0, C_0))) = 0,$$

$$s((X_7, (B_7, C_7)), (X_0, (B_0, C_0))) = \frac{3}{20},$$

$$s((X_8, (B_8, C_8)), (X_0, (B_0, C_0))) = 0,$$

$$s((X_9, (B_9, C_9)), (X_0, (B_0, C_0))) = \frac{113}{150},$$

$$s((X_{10}, (B_{10}, C_{10})), (X_0, (B_0, C_0))) = \frac{151}{300},$$

$$s((X_{11}, (B_{11}, C_{11})), (X_0, (B_0, C_0))) = 0,$$

$$s((X_{12}, (B_{12}, C_{12})), (X_0, (B_0, C_0))) = \frac{43}{50},$$

$$s((X_{13}, (B_{13}, C_{13})), (X_0, (B_0, C_0))) = 0,$$

$$s((X_{14}, (B_{14}, C_{14})), (X_0, (B_0, C_0))) = 0.$$

That is, $(X_{12}, (B_{12}, C_{12})) = (3459, (c, ce))$ with the similarity $s((X_{12}, (B_{12}, C_{12})), (X_0, (B_0, C_0))) = \frac{43}{50}$ is the best granular concept which matches the given clue.

## 5 Numerical experiments

In this section, we conduct some numerical experiments to evaluate the proposed cognitive concept learning algorithms.

In the experiments, we selected five data sets totally from UCI Machine Learning Repository [7] to achieve the evaluation task. That is, Post-Operative Patient data set, Hepatitis data set, Meta-data data set, Breast Cancer Wisconsin (Original) data set and Mushroom data set. Since all of them are not the standard data sets which can directly be used to evaluate the proposed concept learning algorithms, different data pre-processing techniques are required to convert them. In what follows, we introduce the details of these data pre-processing techniques.

(1) Post-Operative Patient data set contains 90 instances (each of them representing a patient) and 8 attributes. The eighth attribute has 3 missing values and we divided its values into three categories: the first category is less than or equal to 7, the second is

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**Table 3** Designing approximate cognitive computing systems for the obtained standard data sets

| ACCS | Design of parameters | |
|---|---|---|
| $\mathcal{AS}^{(1)}$ | $U_1 = \{1 \sim 20\}, A_1 = \{1 \sim 12\},$ | $U_2 = \{1 \sim 45\}, A_2 = \{1 \sim 12\},$ |
| | $U_3 = \{1 \sim 45\}, A_3 = \{1 \sim 25\},$ | $U_4 = \{1 \sim 90\}, A_4 = \{1 \sim 25\}$ |
| $\mathcal{AS}^{(2)}$ | $U_1 = \{1 \sim 40\}, A_1 = \{1 \sim 19\},$ | $U_2 = \{1 \sim 80\}, A_2 = \{1 \sim 19\},$ |
| | $U_3 = \{1 \sim 80\}, A_3 = \{1 \sim 38\},$ | $U_4 = \{1 \sim 155\}, A_4 = \{1 \sim 38\}$ |
| $\mathcal{AS}^{(3)}$ | $U_1 = \{1 \sim 123\}, A_1 = \{1 \sim 11\},$ | $U_2 = \{1 \sim 264\}, A_2 = \{1 \sim 11\},$ |
| | $U_3 = \{1 \sim 264\}, A_3 = \{1 \sim 22\},$ | $U_4 = \{1 \sim 528\}, A_4 = \{1 \sim 22\}$ |
| $\mathcal{AS}^{(4)}$ | $U_1 = \{1 \sim 275\}, A_1 = \{1 \sim 10\},$ | $U_2 = \{1 \sim 350\}, A_2 = \{1 \sim 10\},$ |
| | $U_3 = \{1 \sim 350\}, A_3 = \{1 \sim 20\},$ | $U_4 = \{1 \sim 699\}, A_4 = \{1 \sim 20\}$ |
| $\mathcal{AS}^{(5)}$ | $U_1 = \{1 \sim 203\}, A_1 = \{1 \sim 33\},$ | $U_2 = \{1 \sim 4062\}, A_2 = \{1 \sim 33\},$ |
| | $U_3 = \{1 \sim 4062\}, A_3 = \{1 \sim 45\},$ | $U_4 = \{1 \sim 8124\}, A_4 = \{1 \sim 66\}$ |

**Table 4** Experimental results

| Data set | $|U|$ | $|A|$ | $n$ | Running time(s) | | | |
|---|---|---|---|---|---|---|---|
| | | | | Algorithm 1 | Algorithm 2 | Algorithm 3 | Algorithm 4 |
| Data set 1 | 90 | 25 | 4 | 0.56 | 0.01 | 0.03 | 0.11 |
| Data set 2 | 155 | 38 | 4 | 6.27 | 0.06 | 0.12 | 0.34 |
| Data set 3 | 528 | 22 | 4 | 1.09 | 0.02 | 0.07 | 0.15 |
| Data set 4 | 699 | 20 | 4 | 6.16 | 0.78 | 0.99 | 1.92 |
| Data set 5 | 8,124 | 66 | 4 | 1888.23 | 0.79 | 1.31 | 2.19 |

located between 7 and 14, and the third is greater than or equal to 14. In addition, each of other attributes has three values except the third one having four values. Then, the scaling approach [8] was applied to the data set for generating a standard one. We denote it by Data set 1.

(2) Hepatitis data set has 155 instances and 19 attributes. In this data set, the fifth, sixth and seventh attributes have one missing value, the sixteenth has 4 missing values, the tenth, eleventh, twelfth and thirteenth have 5 missing values, the fourteenth has 6 missing values, the eighth has 9 missing values, the ninth has 10 missing values, the seventeenth has 16 missing values, the fifteenth has 27 missing values, and the eighteenth has 62 missing vales. In the experiments, each attribute was binarized. Then, a standard data set is obtained, and we denote it by Data set 2.

(3) Meta-data data set contains 528 instances and 22 attributes. Since there are huge differences among the attributes, we only selected eleven of them which are from the fourth to the fourteenth. In this data set, both the eleventh and thirteenth attributes have 38 missing values. For the purpose of generating a standard data set, each attribute was binarized. We denote this standard data set by Data set 3.

(4) Breast Cancer Wisconsin (Original) data set contains 699 instances (each of them representing a patient) and 10 attributes. In this data set, the sixth attribute has 16 missing values. Once again, each attribute was binarized to generate a standard data set which is denoted by Data set 4.

(5) Mushroom data set consists of 8124 instances and 22 attributes. The twelfth attribute contains 2480 missing values. In the experiments, we split the values of each attribute, from small to large, into three pairwise disjoint intervals whose lengths are the same. Then, it was converted by the scaling approach into a standard data set. We denote it by Data set 5.

Furthermore, Data sets 1, 2, 3, 4 and 5 were divided into segments for designing their corresponding approximate cognitive computing systems: $\mathcal{AS}^{(1)} = \bigcup_{i=2}^{4}\{\mathcal{AS}^{(1)}_{f_i g_i}\}$, $\mathcal{AS}^{(2)} = \bigcup_{i=2}^{4}\{\mathcal{AS}^{(2)}_{f_i g_i}\}$, $\mathcal{AS}^{(3)} = \bigcup_{i=2}^{4}\{\mathcal{AS}^{(3)}_{f_i g_i}\}$, $\mathcal{AS}^{(4)} = \bigcup_{i=2}^{4}\{\mathcal{AS}^{(4)}_{f_i g_i}\}$ and $\mathcal{AS}^{(5)} = \bigcup_{i=2}^{4}\{\mathcal{AS}^{(5)}_{f_i g_i}\}$. See Table 3 for the details, where ACCS is the abbreviation of "Approximate cognitive computing system". In the table, $U_i = \{x \sim y\}$ means that $U_i$ is constituted by the objects between the $x$-th and $y$-th including the endpoints, so is $A_i$.

For convenience of description, we denote by Algorithms 2, 3, and 4 respectively the processes of learning granular concepts from an object set, a pair of attribute sets, and both of them. Moreover, we took the weight parameters $\lambda = \mu = 0.6$. Then, Algorithms 1, 2, 3 and 4 were applied to Data sets 1, 2, 3, 4 and 5. The corresponding

running time is reported in Table 4, where |U| is the cardinality of the object set, |A| is that of the attribute set, and $n$ is the number of approximate cognitive computing states. Note that in the experiments, we generated 100 clues randomly for Algorithm 2 as well as Algorithms 3 and 4. In other words, only the average running time of Algorithms 2, 3 and 4 is reported in Table 4. Finally, it can be observed from the table that all the algorithms are reasonably efficient for the five chosen data sets.

It deserves to be mentioned that the time complexity of Algorithm 1 is $O(n(|U_n| + |A_n|)|U_n||A_n|)$, while those of Algorithms 2, 3 and 4 are $O(|U_n|^2 + |U_n||A_n| + |A_n|^2)$. That is to say, the time complexity of Algorithm 1 is far more than those of Algorithms 2, 3 and 4 in theory. This was also confirmed indirectly by Data set 5 in the experiments.

## 6 Final remarks

Cognitive concept learning has become a hot topic in recent years, and it has attracted much attention from the communities of formal concept analysis, granular computing and cognitive computing. However, it is still at the immature stage. That is, more theoretical frameworks, effective methods and potential applications are needed to be improved.

Our current work mainly focuses on cognitive concept learning from incomplete information. Specifically, we have proposed a pair of approximate cognitive operators and an approximate cognitive computing system to form and update granular concepts. Moreover, cognitive processes have been simulated based on three types of similarities to learn granular concept from a given clue. In addition, we have conducted some numerical experiments to evaluate the effectiveness of the proposed concept learning methods.

Compared to the existing work, the main contributions of our paper are as follows: (1) new cognitive operators have been defined for incomplete information environment; (2) an approximate cognitive computing system has been designed for applying incomplete information fusion; (3) three types of similarities have been presented and used to learn granular concepts from given clues.

It should also be pointed out that our work is completely different from the one in [13] although both of them discussed concept learning from incomplete information. As a matter of fact, our work is to learn part of concepts, while the one in [13] is to learn all of concepts. In addition, concept learning in our work was realized by an axiomatic method, while that in [13] was done by a constructive method.

Note that learning cognitive concepts from incomplete information is a challenging task. Although our work has put forward some theoretical frameworks and methods to address this problem, it is still not enough in many aspects. For example, how to apply them in the real world? It includes the semantic explanation of approximate cognitive concepts, the assignment of the parameters $\lambda$ and $\mu$, the evaluation of the learnt granular concepts, and how to improve the learning efficiency [17]. Moreover, cognitive logic [26, 50] should be incorporated into cognitive concept learning, and uncertainty [29] needs to be considered in incomplete information [29]. Besides, learning cognitive concepts from fuzzy data [31–33] also deserves to be investigated. Undoubtedly, approaches to cognitive concept learning from incomplete information cannot be directly extended to the case of fuzzy information since both knowledge representation and information measure are extremely different [1, 9, 30]. These issues will be studied in our future work.

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